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Forum of Mathematics, Sigma / Volume 2 / April 2014 / e2  
DOI: 10.1017/fms.2013.6, Published online: 10 March 2014

**Link to this article:** [http://journals.cambridge.org/abstract\\_S2050509413000066](http://journals.cambridge.org/abstract_S2050509413000066)

### How to cite this article:

YEMON CHOI, ILIJAS FARAH and NARUTAKA OZAWA (2014). A NONSEPARABLE AMENABLE OPERATOR ALGEBRA WHICH IS NOT ISOMORPHIC TO A  $C^*$ -ALGEBRA . Forum of Mathematics, Sigma, 2, e2  
doi:10.1017/fms.2013.6

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# A NONSEPARABLE AMENABLE OPERATOR ALGEBRA WHICH IS NOT ISOMORPHIC TO A $C^*$ -ALGEBRA

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Received 29 September 2013; accepted 28 November 2013

## Abstract

It has been a long-standing question whether every amenable operator algebra is isomorphic to a (necessarily nuclear)  $C^*$ -algebra. In this note, we give a nonseparable counterexample. Finding out whether a separable counterexample exists remains an open problem. We also initiate a general study of unitarizability of representations of amenable groups in  $C^*$ -algebras and show that our method cannot produce a separable counterexample.

2010 Mathematics Subject Classification: 47L30, (primary); 46L05, 03E75 (secondary)

## 1. Introduction

The notion of amenability for Banach algebras was introduced by Johnson [Jo72] in the 1970s and has been studied intensively since then (see a more recent monograph [Ru02]). For several natural classes of Banach algebras, the amenability property is known to single out the ‘good’ members of those classes. For example, Johnson’s fundamental observation [Jo72] is that the Banach algebra  $L^1(G)$  of a locally compact group  $G$  is amenable if and only if the group

$G$  is amenable. Another example is the celebrated result of Connes [Co78] and Haagerup [Ha83] which states that a  $C^*$ -algebra is amenable as a Banach algebra if and only if it is nuclear.

In this paper, we are interested in the class of *operator algebras*. By an operator algebra, we mean a (not necessarily self-adjoint) norm-closed subalgebra of  $\mathbb{B}(H)$ , the  $C^*$ -algebra of the bounded linear operators on a Hilbert space  $H$ . It has been asked by several researchers whether every amenable operator algebra is isomorphic to a (necessarily nuclear)  $C^*$ -algebra. The problem has been solved affirmatively in several special cases: for subalgebras of commutative  $C^*$ -algebras [Še77], and subsequently for operator algebras generated by normal elements [CL95]; for subalgebras of compact operators [Gi06, Wi95]; for 1-amenable operator algebras [BL04, Theorem 7.4.18]; and for commutative subalgebras of finite von Neumann algebras [Ch13].

Here we give the first counterexample to the above problem. In fact, our counterexample is a subalgebra of the homogeneous  $C^*$ -algebra  $\ell_\infty(\mathbb{N}, \mathbb{M}_2)$ . Hence the result of [Še77] is actually quite sharp and the result of [Ch13] does not generalize to an arbitrary subalgebra of a finite von Neumann algebra.

**THEOREM 1.** *There is a unital amenable operator algebra  $\mathcal{A}$  which is not isomorphic to a  $C^*$ -algebra. The algebra  $\mathcal{A}$  is a subalgebra of  $\ell_\infty(\mathbb{N}, \mathbb{M}_2)$  with density character  $\aleph_1$ , and is an inductive limit of unital separable subalgebras  $\{\mathcal{A}_i\}_{i < \aleph_1}$ , each of which is conjugated to a  $C^*$ -subalgebra of  $\ell_\infty(\mathbb{N}, \mathbb{M}_2)$  by an invertible element  $v_i \in \ell_\infty(\mathbb{N}, \mathbb{M}_2)$ , such that  $\sup_i \|v_i\| \|v_i^{-1}\| < \infty$ . Moreover, for any  $\varepsilon > 0$ , one can choose  $\mathcal{A}$  to be  $(1 + \varepsilon)$ -amenable.*

Here,  $C$ -amenable means that the amenability constant is at most  $C$  (see [Ru02, Definition 2.3.15]). One drawback of our counterexample is that it is inevitably nonseparable, as explained by Theorem 8 below, and the existence of a separable counterexample remains an open problem. We note that if such an example exists, then there is one among the subalgebras of the finite von Neumann algebra  $\prod_{n=1}^{\infty} \mathbb{M}_n$ . Indeed, by Voiculescu's theorem [Vo91], the cone  $C_0((0, 1], \mathcal{A})$  of a separable operator algebra  $\mathcal{A}$  can be realized as a closed subalgebra of  $\prod_{n=1}^{\infty} \mathbb{M}_n / \bigoplus_{n=1}^{\infty} \mathbb{M}_n$ . The cone of  $\mathcal{A}$  is amenable (see [Ru02, Exercise 2.3.6]), and its preimage  $\tilde{\mathcal{A}}$  in  $\prod_{n=1}^{\infty} \mathbb{M}_n$  is an extension of the cone by the amenable algebra  $\bigoplus_{n=1}^{\infty} \mathbb{M}_n$ ; hence  $\tilde{\mathcal{A}}$  is amenable (see [Ru02, Theorem 2.3.10]).  $\tilde{\mathcal{A}}$  is not isomorphic to a  $C^*$ -algebra, since it has  $\mathcal{A}$  as a quotient and every closed two-sided ideal in a  $C^*$ -algebra is automatically  $*$ -closed.

*Note added in proof.* In a recent preprint (<http://arxiv.org/abs/1311.2982>), L. W. Marcoux and A. Popov have proved that every abelian, amenable operator

algebra is similar to an abelian  $C^*$ -algebra. This subsumes the results of [Wi95, Ch13].

## 2. Proof of Theorem 1

Let  $\mathcal{C}$  be a unital  $C^*$ -algebra,  $\Gamma$  be a group, and  $\pi : \Gamma \rightarrow \mathcal{C}$  be a representation, that is,  $\pi(s)$  is invertible for every  $s \in \Gamma$  and  $\pi(st) = \pi(s)\pi(t)$  for every  $s, t \in \Gamma$ . The representation  $\pi$  is said to be *uniformly bounded* if  $\|\pi\| := \sup_s \|\pi(s)\| < +\infty$ . It is said to be *unitarizable* if there is an invertible element  $v$  in  $\mathcal{C}$  such that  $\text{Ad}_v \circ \pi$  is a unitary representation. Here  $\text{Ad}_v(c) = vc v^{-1}$  for  $c \in \mathcal{C}$ . The element  $v$  is called a *similarity* element. A well-known theorem of Szökefalvi-Nagy, Day, Dixmier, and of Nakamura and Takeda, states that every uniformly bounded representation of an amenable group  $\Gamma$  into a von Neumann algebra is unitarizable. In fact the latter property characterizes amenability by Pisier's theorem [Pi07]. In particular, the operator algebra  $\overline{\text{span}} \pi(\Gamma)$  generated by a uniformly bounded representation  $\pi$  of an amenable group  $\Gamma$  is an amenable operator algebra which is isomorphic to a nuclear  $C^*$ -algebra. See [Pi01, Ru02] for general information about uniformly bounded representations and amenable Banach algebras, respectively.

Let us fix the notation. Let  $\mathbb{M}_2$  be the 2-by-2 full matrix algebra,  $\ell_\infty(\mathbb{N}, \mathbb{M}_2)$  be the  $C^*$ -algebra of the bounded sequences in  $\mathbb{M}_2$ , and  $c_0(\mathbb{N}, \mathbb{M}_2)$  be the ideal of the sequences that converge to zero. We shall freely identify  $\ell_\infty(\mathbb{N}, \mathbb{M}_2)$  with  $\ell_\infty(\mathbb{N}) \otimes \mathbb{M}_2$ , and  $\ell_\infty(\mathbb{N}, \mathbb{M}_2)/c_0(\mathbb{N}, \mathbb{M}_2)$  with  $\mathcal{C}(\mathbb{N}) \otimes \mathbb{M}_2$ , where  $\mathcal{C}(\mathbb{N}) = \ell_\infty(\mathbb{N})/c_0(\mathbb{N})$ . The quotient map from  $\ell_\infty(\mathbb{N})$  (or  $\ell_\infty(\mathbb{N}) \otimes \mathbb{M}_2$ ) onto  $\mathcal{C}(\mathbb{N})$  (or  $\mathcal{C}(\mathbb{N}) \otimes \mathbb{M}_2$ ) is denoted by  $Q$ .

LEMMA 2. *Let  $\Gamma$  be an abelian group and  $\pi : \Gamma \rightarrow \mathcal{C}(\mathbb{N}) \otimes \mathbb{M}_2$  be a uniformly bounded representation. Then the amenable operator algebra*

$$\mathcal{A} := Q^{-1}(\overline{\text{span}} \pi(\Gamma)) \subset \ell_\infty(\mathbb{N}, \mathbb{M}_2)$$

*is isomorphic to a  $C^*$ -algebra if and only if  $\pi$  is unitarizable.*

*Proof.* First of all, we observe that the operator algebra  $\mathcal{A}$  is indeed amenable because it is an extension of an amenable Banach algebra  $\overline{\text{span}} \pi(\Gamma)$  by the amenable Banach algebra  $c_0(\mathbb{N}, \mathbb{M}_2)$  (see [Ru02, Theorem 2.3.10]). Suppose now that  $\pi$  is unitarizable and  $v \in \mathcal{C}(\mathbb{N}) \otimes \mathbb{M}_2$  has the property that  $\text{Ad}_v \circ \pi$  is unitary. We may assume that  $v$  is positive, by taking the positive component from its polar decomposition. Since  $v$  is invertible, we can choose a representing sequence  $v_m$ , for  $m \in \mathbb{N}$  of  $v$  such that each  $v_m$  is positive and moreover  $1/\|v^{-1}\| \leq v_m \leq \|v\|$  for all  $m$ . In particular each  $v_m$  is invertible and  $\|v_m\| \|v_m^{-1}\| \leq \|v\| \|v^{-1}\|$  for all

*m.* Now we have a representing sequence of an invertible lift  $\tilde{v} \in \ell_\infty(\mathbb{N}, \mathbb{M}_2)$  of  $v$  such that  $\|\tilde{v}\| \|\tilde{v}^{-1}\| = \|v\| \|v^{-1}\|$ . Then  $\tilde{v} \mathcal{A} \tilde{v}^{-1} = Q^{-1}(\overline{\text{span}}(\text{Ad}_v \circ \pi(\Gamma)))$  is a self-adjoint  $C^*$ -subalgebra of  $\ell_\infty(\mathbb{N}, \mathbb{M}_2)$ . Conversely, suppose that  $\mathcal{A}$  is isomorphic to a  $C^*$ -algebra, which is necessarily nuclear. Then thanks to the solution of Kadison's similarity problem for nuclear  $C^*$ -algebras (see [Pi01, Theorem 7.16] or [Pi07, Theorem 1]), there is a  $\tilde{v}$  in the von Neumann algebra  $\ell_\infty(\mathbb{N}, \mathbb{M}_2)$  such that  $\tilde{v} \mathcal{A} \tilde{v}^{-1}$  is a  $C^*$ -subalgebra. Let  $v = Q(\tilde{v}) \in \mathcal{C}(\mathbb{N}) \otimes \mathbb{M}_2$ . Since  $Q(\tilde{v} \mathcal{A} \tilde{v}^{-1})$  is a commutative  $C^*$ -subalgebra of  $\mathcal{C}(\mathbb{N}) \otimes \mathbb{M}_2$ , for every  $s \in \Gamma$ , the element  $v\pi(s)v^{-1}$  is normal with its spectrum in the unit circle, which implies that  $v\pi(s)v^{-1}$  is unitary.  $\square$

The above proof uses the fact that every (not necessarily separable) amenable  $C^*$ -algebra is nuclear, as well as the solution to Kadison's similarity problem for nuclear  $C^*$ -algebras. The reader may appreciate a more elementary and self-contained proof. Assume that  $\theta$  is a bounded homomorphism of a unital  $C^*$ -algebra  $\mathcal{A}$  into  $\ell_\infty(\mathbb{N}, \mathbb{M}_2)$ . We need to prove that  $\theta$  is similar to a  $*$ -homomorphism. It suffices to show that every coordinate map is similar to a  $*$ -homomorphism and that the similarities are implemented by a uniformly bounded sequence  $v_n$ , for  $n \in \mathbb{N}$ , of operators. Consider the restriction of  $\theta$  to the unitary group  $G$  of  $\mathcal{A}$ . At the  $n$ th coordinate we have a bounded homomorphism from  $G$  to  $\text{GL}(2, \mathbb{C})$ . Since a bounded subgroup of  $\text{GL}(2, \mathbb{C})$  is included in a compact subgroup, by a standard averaging argument we find  $v_n$  such that  $\text{Ad}_{v_n} \circ \theta$  is a unitary representation of  $G$ . The operators  $v_n$  are easily seen to satisfy the required properties.

*Proof of Theorem 1.* We consider two 2-by-2 order-two invertible matrices which are not simultaneously unitarizable. For instance, let  $s^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  and  $s^1 = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$ . Then by compactness, one has

$$\varepsilon(C) := \inf\{d(vs^0v^{-1}, \mathcal{U}) + d(vs^1v^{-1}, \mathcal{U}) : v \in \mathbb{M}_2^{-1}, \|v\| \|v^{-1}\| \leq C\} > 0$$

for every  $C > 0$ . Here  $\mathcal{U}$  denotes the unitary group of  $\mathbb{M}_2$ .

We shall need two families  $\{E_i^0 : i \in \aleph_1\}$  and  $\{E_i^1 : i \in \aleph_1\}$  of subsets of  $\mathbb{N}$  such that: (i)  $E_i^k \cap E_j^l$  is finite whenever  $(i, k) \neq (j, l)$ ; and (ii) these two families are not *separated*, in the sense that there is no  $F \subseteq \mathbb{N}$  such that both  $E_i^0 \setminus F$  and  $E_i^1 \cap F$  are finite for all  $i$ . The existence of such pair of families follows from [Lu47]. Luzin actually proved much more: he constructed a single family  $\{E_i : i < \aleph_1\}$  of infinite subsets of  $\mathbb{N}$  such that: (i)  $E_i \cap E_j$  is finite whenever  $i \neq j$ ; and (ii) whenever  $X \subseteq \aleph_1$  is such that both  $X$  and  $\aleph_1 \setminus X$  are uncountable, then the families  $\{E_i : i \in X\}$  and  $\{E_i : i \in \aleph_1 \setminus X\}$  cannot be separated (see appendix B below for Luzin's proof).

The projections  $p_i^k = Q(1_{E_i^k}) \in \mathcal{C}(\mathbb{N})$  are mutually orthogonal. For each pair  $(i, k)$ , we define  $s_i^k$  in  $\mathcal{C}(\mathbb{N}) \otimes \mathbb{M}_2$  by

$$s_i^k = p_i^k \otimes s^k + (1 - p_i^k) \otimes 1.$$

Let  $\Gamma := \bigoplus_{i \in \mathbb{N}_1, k \in \{0,1\}} \mathbb{Z}/2\mathbb{Z}$  and  $\{e_i^k\}$  be its standard basis. Then the map  $e_i^k \mapsto s_i^k$  extends to a uniformly bounded representation  $\pi: \Gamma \rightarrow \mathcal{C}(\mathbb{N}) \otimes \mathbb{M}_2$  such that  $\|\pi\| = \max\{\|s^0\|, \|s^1\|\}$ . We claim that  $\pi$  is not unitarizable. Suppose for a contradiction that there is an invertible element  $v \in \mathcal{C}(\mathbb{N}) \otimes \mathbb{M}_2$  such that  $\text{Ad}_v \circ \pi$  is unitary. As in the proof of Lemma 2 we may assume that  $v$  is positive and find a representing sequence  $v_m$ , for  $m \in \mathbb{N}$ , of an invertible lift of  $v$  such that  $\|v_m\| \|v_m^{-1}\| \leq \|v\| \|v^{-1}\|$  for all  $m$ . Let  $\varepsilon = \varepsilon(\|v\| \|v^{-1}\|)$ .

Now let  $F^0 := \{m : d(v_m s^0 v_m^{-1}, \mathcal{U}) < \varepsilon/2\}$ , and note that this set is disjoint from  $F^1 := \{m : d(v_m s^1 v_m^{-1}, \mathcal{U}) < \varepsilon/2\}$ . Therefore we have  $i$  such that  $E_i^0 \setminus F^0$  is infinite or such that  $E_i^1 \setminus F^1$  is infinite. If the former case applies, then

$$\limsup_{n \in E_i^0, n \rightarrow \infty} d(v_n s^0 v_n^{-1}, \mathcal{U}) \geq \varepsilon/2,$$

contradicting the assumption that  $v$  unitarizes  $\pi$ . The case where  $E_i^1 \setminus F^1$  is infinite similarly leads to a contradiction. Thus, by Lemma 2, the preimage of  $\overline{\text{span}} \pi(\Gamma)$  in  $\ell_\infty(\mathbb{N}, \mathbb{M}_2)$  is an amenable operator algebra which is not isomorphic to a  $C^*$ -algebra. Its density character is equal to  $\aleph_1 = |\Gamma|$ .

Let  $\Gamma_i$  be a countable subgroup of  $\Gamma$  and denote the separable algebra  $Q^{-1}(\overline{\text{span}} \pi(\Gamma_i))$  by  $\mathcal{A}_i$ . Theorem 8 below shows that  $\mathcal{A}_i$  is similar inside  $\ell_\infty(\mathbb{N}, \mathbb{M}_2)$  to an amenable  $C^*$ -algebra, with a similarity element  $v_i$  satisfying  $\|v_i\| \|v_i^{-1}\| \leq \|\pi\|^2$ . Furthermore, since every amenable  $C^*$ -algebra is 1-amenable by results of Haagerup [Ha83],  $\mathcal{A}_i$  is  $\|\pi\|^4$ -amenable. Now  $\mathcal{A}$  is the inductive limit of the family  $(\mathcal{A}_i)$  as  $\Gamma_i$  varies over all countable subgroups of  $\Gamma$ . Since each  $\mathcal{A}_i$  is  $\|\pi\|^4$ -amenable, a routine argument with approximate diagonals shows that  $\mathcal{A}$  is also  $\|\pi\|^4$ -amenable: for details see [Ru02, Proposition 2.3.17].

Finally, we explain how our example can be modified to have arbitrarily small amenability constant. For  $0 < t < 1$ , we keep  $s^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  but replace  $s^1$  with  $s^1(t) = \begin{bmatrix} 1 & 0 \\ t & -1 \end{bmatrix}$  in our original construction. Denoting the resulting algebra by  $\mathcal{A}(t)$ , the previous arguments show that  $\mathcal{A}(t)$  is  $\|s^1(t)\|^4$ -amenable, and  $\|s^1(t)\|$  can be made arbitrarily close to 1.  $\square$

We note that a set-theoretical study of the cohomological nature of gaps similar to Luzin's was initiated in [Ta95].

### 3. Unitarizability of uniformly bounded representations

In this section, we develop a general study of (non)unitarizability. First, we shall deal with separable  $C^*$ -algebras. Let  $\mathcal{A}$  be a unital  $C^*$ -algebra and  $\theta$  be a  $*$ -automorphism on  $\mathcal{A}$ . An element  $a \in \mathcal{A}$  is called a *cocycle* if it satisfies

$$\|a\| := \sup_{n \geq 1} \left\| \sum_{k=0}^{n-1} \theta^k(a) \right\| < +\infty.$$

It is *inner* (or a coboundary) if there is  $x \in \mathcal{A}$  such that  $a = x - \theta(x)$ . We recall that the *first bounded cohomology group* (see [Mo01]) of the  $\mathbb{Z}$ -module  $(\mathcal{A}, \theta)$  is defined as

$$H_b^1(\mathcal{A}, \theta) = \{\text{cocycles}\} / \{\text{inner cocycles}\}.$$

When  $\mathcal{A}$  is abelian and  $\theta$  corresponds to a minimal homeomorphism of its spectrum then  $H_b^1$  is trivial (see [Or00, Theorem 2.6]).

We note that every cocycle is approximately inner. Indeed, since  $a_n := \sum_{k=0}^{n-1} \theta^k(a)$  satisfies  $a_{n+1} = a + \theta(a_n)$ , the element  $x_n := n^{-1} \sum_{m=1}^n a_m$  satisfies  $\|x_n\| \leq \|a\|$  and  $\|a - (x_n - \theta(x_n))\| \leq 2n^{-1}\|a\|$ . Suppose for a moment that  $\theta$  is inner,  $\theta = \text{Ad}_u$  for a unitary element  $u \in \mathcal{A}$ , and  $a \in \mathcal{A}$  is a cocycle. Then,  $t = \begin{bmatrix} u & au \\ 0 & u \end{bmatrix}$  is an invertible element in  $\mathbb{M}_2(\mathcal{A})$  such that  $t^n = \begin{bmatrix} u^n & a_n u^n \\ 0 & u^n \end{bmatrix}$  for  $n \geq 1$ . Therefore  $\sup_{n \in \mathbb{Z}} \|t^n\| \leq 1 + \|a\|$  and  $t$  gives rise to a uniformly bounded representation  $\pi_a$  of  $\mathbb{Z}$  into  $\mathcal{A}$ .

LEMMA 3. *Let  $\mathcal{A}$ ,  $u$ ,  $a$ , and  $\pi_a$  be as above. Then the uniformly bounded representation  $\pi_a$  is unitarizable if and only if  $a$  is inner.*

See [Pi01, Lemma 4.5] or [MO10] for the proof of this lemma.

PROPOSITION 4. *Let  $\mathcal{A}$  be a unital separable  $C^*$ -algebra and  $\theta$  be a  $*$ -automorphism of  $\mathcal{A}$ . Suppose that there are a (nonunital)  $\theta$ -invariant  $C^*$ -subalgebra  $\mathcal{A}_0$ , a state  $\phi$  on  $\mathcal{A}_0$ , and a sequence of natural numbers  $n(k)$  such that  $(\phi \circ \theta^{n(k)})_{k=1}^\infty$  converges to 0 pointwise on  $\mathcal{A}_0$ . Then,  $H_b^1(\mathcal{A}, \theta) \neq 0$ .*

*Proof.* By a standard Hahn–Banach convexity argument, we construct an approximate unit  $(h_n)_{n=0}^\infty$  of  $\mathcal{A}_0$  such that  $0 \leq h_n \leq 1$ ,  $h_{n+1}h_n = h_n$ , and  $\|h_n - \theta(h_n)\| < 2^{-n}$  for all  $n$ . We note that  $\phi'(h_n) \rightarrow 1$  for any state  $\phi'$  on  $\mathcal{A}_0$ . Taking a state extension, we may assume that  $\phi$  is defined on  $\mathcal{A}$ . Since  $\mathcal{A}$  is separable, passing to a subsequence, we may assume that  $\phi^k := \phi \circ \theta^{n(k)}$  converges pointwise to a state, say  $\psi$ , on  $\mathcal{A}$ .

Set  $k(1) = 1$ . By induction, one can find strictly increasing sequences  $(m(j))_{j=1}^\infty$  and  $(k(j))_{j=1}^\infty$  of natural numbers such that  $\phi^{k(i)}(h_{m(j)}) > 1 - 2^{-j}$  for every  $i \leq j$

and  $\phi^{k(j+1)}(h_{m(j)}) < 2^{-j}$  for every  $j$ . Let

$$x = \text{SOT-}\sum_{j=1}^{\infty} (h_{m(2j)} - h_{m(2j-1)}) \in \mathcal{A}^{**}.$$

We extend  $\theta$  and  $\phi$  on  $\mathcal{A}^{**}$  by ultraweak continuity. One has  $a := x - \theta(x) \in \mathcal{A}$ , since it is a norm-convergent series in  $\mathcal{A}_0$ . By a telescoping argument,  $a$  is a cocycle.

Suppose for the sake of obtaining a contradiction that  $a$  is inner and  $x - \theta(x) = y - \theta(y)$  for some  $y \in \mathcal{A}$ . Then,  $y \in \mathcal{A}$  and  $\theta(x - y) = x - y$ . It follows that  $\phi^{k(j)}(y) \rightarrow \psi(y)$  and  $\phi^{k(j)}(x - y) = \phi(x - y)$ . Hence the sequence  $(\phi^{k(j)}(x))_{j=1}^{\infty}$  converges. However, for  $j \geq 1$ ,

$$\phi^{k(2j)}(x) \geq \phi^{k(2j)}(h_{m(2j)} - h_{m(2j-1)}) \geq 1 - \frac{1}{2^{2j}} - \frac{1}{2^{2j-1}}$$

and

$$\phi^{k(2j+1)}(x) \leq \phi^{k(2j+1)}\left(\sum_{i=1}^j h_{m(2i)}\right) + \sum_{i=j+1}^{\infty} (1 - \phi^{k(2j+1)}(h_{m(2i-1)})) \leq \frac{1}{4}.$$

Hence, the sequence  $(\phi^{k(j)}(x))_{j=1}^{\infty}$  does not converge, and we have a contradiction.  $\square$

Examples of  $\mathcal{A}_0$ ,  $\phi$  and  $\theta$  as in the statement of Proposition 4 are the ideal  $\mathbb{K}$  of compact operators on  $\mathbb{B}(\ell_2(\mathbb{Z}))$ , any one of its states, and the bilateral shift on  $\ell_2(\mathbb{Z})$ .

**LEMMA 5.** *For every unital separable  $C^*$ -algebra  $\mathcal{A}$  which is not of type I, there is a unitary element  $u \in \mathcal{A}$  such that  $H_b^1(\mathcal{A}, \text{Ad}_u) \neq 0$ .*

*Proof.* Let  $z$  be the bilateral shift on  $\ell_2(\mathbb{Z})$  and take a self-adjoint element  $h \in \mathbb{B}(\ell_2(\mathbb{Z}))$  such that  $z = \exp(\sqrt{-1}h)$ . Let  $\mathcal{C} \subset \mathbb{B}(\ell_2(\mathbb{Z}))$  be the unital  $C^*$ -subalgebra generated by  $\mathbb{K}$  and  $h$ , and let  $\phi_0$  be the vector state at  $\delta_0$ . Since  $\mathcal{C}$  is an extension of a commutative  $C^*$ -algebra by  $\mathbb{K}$ , it is nuclear. By Kirchberg's theorem and Glimm's theorem in tandem [Ki95, Corollary 1.4(vii)], there are a unital  $C^*$ -subalgebra  $\mathcal{A}_1$  of  $\mathcal{A}$  and a surjective  $*$ -homomorphism  $\pi$  from  $\mathcal{A}_1$  onto  $\mathcal{C}$ . Let  $g \in \mathcal{A}_1$  be a self-adjoint lift of  $h$  and let us have  $u := \exp(\sqrt{-1}g) \in \mathcal{A}_1$ , which is a unitary lift of  $z$ . Then  $\mathcal{A}_0 = \pi^{-1}(\mathbb{K})$  is an  $\text{Ad}_u$ -invariant subalgebra and the state  $\phi = \phi_0 \circ \pi$  satisfies  $\phi \circ (\text{Ad}_u)^n \rightarrow 0$  pointwise on  $\mathcal{A}_0$ . Hence the result follows from Proposition 4.  $\square$

Combining Lemmas 5 and 3, we arrive at the following theorem.



**THEOREM 6.** *For every unital separable  $C^*$ -algebra  $\mathcal{A}$  which is not of type I, there is a uniformly bounded representation of  $\mathbb{Z}$  into  $\mathbb{M}_2(\mathcal{A})$  which is not unitarizable.*

Now, we shall deal with nonseparable  $C^*$ -algebras. Our approach uses model theory of metric structures and the extension of Pedersen's techniques [Pe88] as presented in [FH13]. The following is [FH13, Definition 1.1], with a misleading typo corrected.

**DEFINITION 7.** Given a  $C^*$ -algebra  $\mathcal{M}$ , a *degree-one  $*$ -polynomial* with coefficients in  $\mathcal{M}$  is a linear combination of terms of the form  $axb$ ,  $ax^*b$  and  $a$  with  $a, b$  in  $\mathcal{M}$ . A  $C^*$ -algebra  $\mathcal{M}$  is said to be *countably degree-one saturated* if for every countable family of degree-one  $*$ -polynomials  $P_n(\bar{x})$  with coefficients in  $\mathcal{M}$  and variables  $x_m$ , for  $m \in \mathbb{N}$ , and every family of compact sets  $K_n \subset \mathbb{R}$ , for  $n \in \mathbb{N}$ , the following are equivalent (writing  $\bar{b}$  for  $(b_1, b_2, \dots)$  and  $\mathcal{M}_{\leq 1}$  for the closed unit ball of  $\mathcal{M}$ ).

- (1) There are  $b_m \in \mathcal{M}_{\leq 1}$ , for  $m \in \mathbb{N}$ , such that  $\|P_n(\bar{b})\| \in K_n$  for all  $n$ .
- (2) For every  $N \in \mathbb{N}$  there are  $b_m \in \mathcal{M}_{\leq 1}$ , for  $m \in \mathbb{N}$ , such that

$$\text{dist}(\|P_n(\bar{b})\|, K_n) \leq \frac{1}{N}$$

for all  $n \leq N$ .

A type  $\{P_n(\bar{x}) \in K_n : n \in \mathbb{N}\}$  satisfying (1) is said to be *realized* in  $\mathcal{M}$  and a type satisfying (2) is said to be *consistent* with (or *approximately finitely realized in*)  $\mathcal{M}$ . Coronas of  $\sigma$ -unital  $C^*$ -algebras, in particular the Calkin algebra  $\mathcal{Q}(\ell_2)$  and  $\mathcal{C}(\mathbb{N}) \otimes \mathbb{M}_2$ , as well as ultraproducts associated with nonprincipal ultrafilters on  $\mathbb{N}$ , are countably degree-one saturated [FH13, Theorem 1.4]. In each of these cases, given a consistent type, a realization  $\bar{b}$  is assembled from the approximate realizations  $\bar{b}^n$ , for  $n \in \mathbb{N}$ , and a carefully chosen, appropriately quasicontral approximate unit  $e_n$ , for  $n \in \mathbb{N}$ , as  $\bar{b} = \sum_n (e_n - e_{n+1})^{1/2} \bar{b}^n (e_n - e_{n+1})^{1/2}$ . See [FH13] for details and more examples of countably degree-one saturated  $C^*$ -algebras.

**THEOREM 8.** *Let  $\mathcal{M}$  be a unital countably degree-one saturated  $C^*$ -algebra. Then, every uniformly bounded representation  $\pi : \Gamma \rightarrow \mathcal{M}$  of a countable amenable group  $\Gamma$  into  $\mathcal{M}$  is unitarizable. Moreover a similarity element  $v$  can be chosen such that it satisfies  $\|v\| \|v^{-1}\| \leq \|\pi\|^2$ .*

*Proof.* The proof is analogous to the standard one (see [Pi01, Theorem 0.6]), modulo applying countable degree-one saturation. Consider the type in variable

$x$  over  $\mathcal{M}$  consisting of conditions  $\|x - x^*\| = 0$ ,  $\|x\| \leq \|\pi\|^2$ ,  $\|\|\pi\|^2 - x\| \leq \|\pi\|^2 - \|\pi\|^{-2}$ , and  $\|\pi(s)x\pi(s)^* - x\| = 0$  for all  $s \in \Gamma$ .

We now check that this type is consistent. Let  $(F_n)_{n=1}^\infty$  be a Følner sequence of finite subsets of  $\Gamma$ . Then,

$$h_n = \frac{1}{|F_n|} \sum_{t \in F_n} \pi(t)\pi(t)^*,$$

are positive elements in  $\mathcal{M}$  such that  $\|\pi\|^{-2} \leq h_n \leq \|\pi\|^2$  and

$$\|\pi(s)h_n\pi(s)^* - h_n\| \leq \frac{|F_n \Delta sF_n|}{|F_n|} \|\pi\|^2 \rightarrow 0$$

for every  $s \in \Gamma$ . Hence this type is consistent and by countable degree-one saturation there is  $h \in \mathcal{M}$  which realizes it. Therefore we have  $h = h^*$ ,  $\|h\| \leq \|\pi\|^2$ ,  $\|\|\pi\|^2 - h\| \leq \|\pi\|^2 - \|\pi\|^{-2}$ , and  $\pi(s)h\pi(s)^* = h$  for every  $s \in \Gamma$ . It follows that  $h$  is a positive element such that  $\|\pi\|^{-2} \leq h \leq \|\pi\|^2$  and the invertible elements  $h^{-1/2}\pi(s)h^{1/2}$  satisfy

$$(h^{-1/2}\pi(s)h^{1/2})(h^{-1/2}\pi(s)h^{1/2})^* = h^{-1/2}\pi(s)h\pi(s)^*h^{-1/2} = 1,$$

that is,  $h^{-1/2}\pi(s)h^{1/2}$  are unitary.  $\square$

Theorem 8 shows that the method used in the proof of Theorem 1 cannot be used to produce a separable counterexample.

### Acknowledgements

This joint work was initiated when the second and third authors participated in the workshop ‘C\*-Algebren’ (ID:1335) held at the Mathematisches Forschungsinstitut Oberwolfach in August 2013. We are grateful to the organizers S. Echterhoff, M. Rørdam, S. Vaes, and D. Voiculescu, and the institute for giving the authors an opportunity of producing a joint work. We are also grateful to N. C. Phillips for useful conversations during the workshop and helpful remarks on the first version of this paper, and the third author would like to thank N. Monod and H. Matui for valuable conversations. Finally, we would like to thank S. A. White for his encouragement to include the last sentence of Theorem 1.

The first author was supported by NSERC Discovery Grant 402153-2011. The second author was partially supported by NSERC, a Velux Visiting Professorship, and the Danish Council for Independent Research through Asger Törnquist’s grant No. 10-082689/FNU. The third author was partially supported by JSPS (23540233).

## Appendix A. A correction for [Ch13]

We take the opportunity to fill a small gap in [Ch13]. The main result of that paper is only proved for commutative, amenable subalgebras of  $\sigma$ -finite, finite von Neumann algebras. It is then stated in [Ch13] that the general case follows from the  $\sigma$ -finite one because any finite von Neumann algebra  $\mathcal{M}$  decomposes as a direct product  $\prod_i \mathcal{M}_i$  where each  $\mathcal{M}_i$  is  $\sigma$ -finite. However, the example of the present paper shows that similarity to a  $C^*$ -algebra is not preserved by taking inductive limits, even with a uniform bound on the similarity elements, so more justification is needed. Instead, we may argue as follows. Let  $\mathcal{A}$  be an amenable subalgebra of  $\mathcal{M}$  and let  $\mathcal{A}_i$  be its image under the projection  $\mathcal{M} \rightarrow \mathcal{M}_i$ . Applying the main result of [Ch13] to each  $\mathcal{A}_i$ , we obtain a uniformly bounded family  $v_i \in \mathcal{M}_i$  such that  $v_i \mathcal{A}_i v_i^{-1}$  is a commutative  $C^*$ -subalgebra of  $\mathcal{M}_i$ . Take  $v$  to be the direct product of the  $v_i$ . Then  $v \mathcal{A} v^{-1}$  is an amenable subalgebra of the commutative  $C^*$ -algebra  $\prod_i v_i \mathcal{A}_i v_i^{-1}$ , and hence by [Še77] it is self-adjoint.

## Appendix B. A construction of Luzin's gap

For the reader's convenience we prove Luzin's theorem. Following von Neumann, we identify  $n \in \mathbb{N}$  with the set  $\{0, 1, \dots, n-1\}$ . We construct a family  $E_i$ , for  $i < \aleph_1$ , of infinite subsets of  $\mathbb{N}$  such that:

- (1)  $E_i \cap E_j$  is finite whenever  $i \neq j$ ; and
- (2) for every  $i$  and every  $m \in \mathbb{N}$  the set  $\{j < i : E_j \cap E_i \subseteq m\}$  is finite.

The construction is by recursion. For a finite  $i$  let  $E_i = \{2^i(2k+1) : k \in \mathbb{N}\}$ . Assume that  $i < \aleph_1$  is infinite and the sets  $E_j$ , for  $j < i$ , were chosen to satisfy the requirements. Since  $i$  is countable, we can re-enumerate  $E_j$ , for  $j < i$  as  $F_n$ , for  $n \in \mathbb{N}$ .

Now let  $k(0) = 0$  and  $k(n) = \min F_n \setminus (k(n-1) \cup \bigcup_{l < n} F_l)$  for  $n \geq 1$ . The sequence  $\{k(n)\}$  is strictly increasing and  $k(n) \in F_l$  implies  $n \leq l$ . Therefore  $E_i = \{k(n) : n \in \mathbb{N}\}$  is infinite and  $E_i \cap F_n \subseteq \{k(0), \dots, k(n)\}$  is finite for all  $n$ . Finally, for any  $m \in \mathbb{N}$  the set  $\{n \in \mathbb{N} : F_n \cap E_i \subseteq m\} \subseteq \{n : k(n) < m\}$  is finite.

This describes the recursive construction of a family  $E_i$ , for  $i < \aleph_1$ , satisfying (1) and (2).

We claim that for any  $X \subseteq \aleph_1$  such that  $X$  and  $\aleph_1 \setminus X$  are uncountable the families  $\{E_i : i \in X\}$  and  $\{E_i : i \in \aleph_1 \setminus X\}$  cannot be separated. Assume otherwise, and fix  $F \subseteq \mathbb{N}$  separating them. Since  $E_i \setminus F$  is finite for all  $i \in X$ , there is an  $m \in \mathbb{N}$  such that  $X' = \{i \in X : E_i \setminus F \subseteq m\}$  is uncountable. By increasing  $m$  if necessary we can assure that  $Y' = \{i \in \aleph_1 \setminus X : E_i \cap F \subseteq m\}$  is also uncountable.

Pick  $i \in Y'$  such that  $X'' = \{j \in X' : j < i\}$  is infinite. Then for each  $j \in X''$  we have  $E_j \cap E_i \subseteq (E_j \setminus F) \cup (E_i \cap F) \subseteq m$ . But this contradicts (2).

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